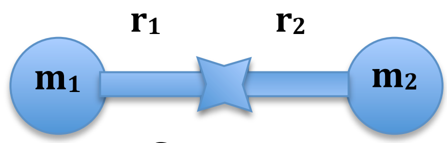
How do we get the reduced mass?



$$Kinetic\ Energy = \frac{m_1}{2}\left(\frac{dr_1}{dt}\right)^2 + \frac{m_2}{2}\left(\frac{dr_2}{dt}\right)^2$$

Let:

$M = m_1 + m_2 =$ total mass

$R = \frac{(m_2 r_2 + m_1 r_1)}{M} =$ center of mass

$r = (r_1 - r_2) =$ intermass distance

What is the kinetic energy of center of mass (CoM)?:

$$CoM\ \ Kinetic\ Energy = \frac{M}{2}\left(\frac{dR}{dt}\right)^2 = \frac{M}{2}\left(\frac{d}{dt}\left[\frac{(m_2 r_2 + m_1 r_1)}{M}\right]\right)^2 = \frac{1}{2M}\left(m_2\left(\frac{dr_2}{dt}\right) + m_1\left(\frac{dr_1}{dt}\right)\right)^2$$

$$= \frac{1}{2M}\left(m_2^2\left(\frac{dr_2}{dt}\right)^2 + 2m_1 m_2\left(\frac{dr_1}{dt}\right)\left(\frac{dr_2}{dt}\right) + m_1^2\left(\frac{dr_2}{dt}\right)^2\right)$$

Let's multiply the actual kinetic energy by M/M:

$$Kinetic\ Energy = \frac{M}{M}\left\{\frac{m_1}{2}\left(\frac{dr_1}{dt}\right)^2 + \frac{m_2}{2}\left(\frac{dr_2}{dt}\right)^2\right\} = \frac{1}{2M}\left\{\frac{Mm_1}{2}\left(\frac{dr_1}{dt}\right)^2 + \frac{Mm_2}{2}\left(\frac{dr_2}{dt}\right)^2\right\}$$

$$= \frac{1}{2M}\left\{(m_1 + m_2)m_1\left(\frac{dr_1}{dt}\right)^2 + (m_1 + m_2)m_2\left(\frac{dr_2}{dt}\right)^2\right\}$$

$$= \frac{1}{2M}\left\{(m_1^2 + m_1 m_2)\left(\frac{dr_1}{dt}\right)^2 + (m_1 m_2 + m_2^2)\left(\frac{dr_2}{dt}\right)^2\right\}$$

$$= \frac{1}{2M}\left\{m_1^2\left(\frac{dr_1}{dt}\right)^2 + m_1 m_2\left[\left(\frac{dr_1}{dt}\right)^2 + \left(\frac{dr_2}{dt}\right)^2\right] + m_2^2\left(\frac{dr_2}{dt}\right)^2\right\}$$

To this we add and subtract $\frac{2m_1 m_2}{2M}\left(\frac{dr_1}{dt}\right)\left(\frac{dr_2}{dt}\right)$:

$$= \frac{1}{2M}\left\{m_1^2\left(\frac{dr_1}{dt}\right)^2 + m_1 m_2\left[\left(\frac{dr_1}{dt}\right)^2 + \left(\frac{dr_2}{dt}\right)^2\right] + m_2^2\left(\frac{dr_2}{dt}\right)^2\right\} + \frac{2m_1 m_2}{2M}\left(\frac{dr_1}{dt}\right)\left(\frac{dr_2}{dt}\right) - \frac{2m_1 m_2}{2M}\left(\frac{dr_1}{dt}\right)\left(\frac{dr_2}{dt}\right)$$

$$= \frac{1}{2M}\left\{m_1^2\left(\frac{dr_1}{dt}\right)^2 + 2m_1 m_2\left(\frac{dr_1}{dt}\right)\left(\frac{dr_2}{dt}\right) + m_2^2\left(\frac{dr_2}{dt}\right)^2\right\} + \frac{m_1 m_2}{2M}\left[\left(\frac{dr_1}{dt}\right)^2 - 2\left(\frac{dr_1}{dt}\right)\left(\frac{dr_2}{dt}\right) + \left(\frac{dr_2}{dt}\right)^2\right]$$

Looking at this, the first half is the kinetic energy of the CoM:

$$= \frac{M}{2}\left(\frac{dR}{dt}\right)^2 + \frac{m_1 m_2}{2M}\left[\left(\frac{dr_1}{dt}\right)^2 - 2\left(\frac{dr_1}{dt}\right)\left(\frac{dr_2}{dt}\right) + \left(\frac{dr_2}{dt}\right)^2\right]$$

The second part is $m_1m_2/2M$ multiplied by:

$$\left(\frac{dr_1}{dt}\right)^2 - 2\left(\frac{dr_1}{dt}\right)\left(\frac{dr_2}{dt}\right) + \left(\frac{dr_2}{dt}\right)^2 = \left[\left(\frac{dr_1}{dt}\right) - \left(\frac{dr_2}{dt}\right)\right]^2 = \left[\frac{d}{dt}(r_1 - r_2)\right]^2 = \left[\frac{dr}{dt}\right]^2$$

We can call m_1m_2/M the "reduced mass" and give it the symbol μ, i.e.:

$$Kinetic\ Energy = \frac{M}{2}\left(\frac{dR}{dt}\right)^2 + \frac{\mu}{2}\left[\frac{dr}{dt}\right]^2$$

Since the CoM does not move in a vibration, the first term on the right is 0, leaving

$$Kinetic\ Energy = \frac{m_1}{2}\left(\frac{dr_1}{dt}\right)^2 + \frac{m_2}{2}\left(\frac{dr_2}{dt}\right)^2 = \frac{\mu}{2}\left[\frac{dr}{dt}\right]^2$$