EE4440 HW#2 Solution

February 2, 2011

1. Answer the following:

   (a) What kind of spectrum does a periodic signal have?
   Solution: Line or discrete.

   (b) What is the procedure (give it step by step with equations) for finding the spectrum of a periodic signal? How does this connect with the Fourier series?
   Solution:

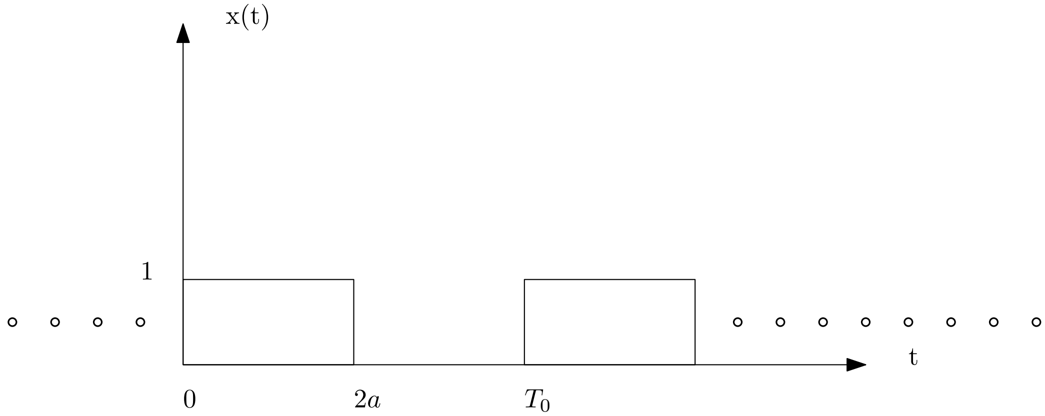
      i. Find the Fourier Series coefficients of the signal ($D_n = \frac{1}{T_0}\int_{T_0} x(t)e^{-jn\omega_0 t}dt$)

      ii. Use the coefficients to write the signal as a Fourier Series ($x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$)

      iii. Take the Fourier Transform of the above expression term by term. Because the FT is linear it operates only on the complex exponential term in the sum. So $X(\omega) = \sum_{n=-\infty}^{\infty} D_n \delta(\omega - n\omega_0)$.

      iv. The FT of a signal is related to the FS by the fact that the FT of a periodic signal is a train of delta functions in frequency who's spacing is dictated by the fundamental frequency, and who's magnitude is dictated by the FS coefficients.

2. For the pulse train depicted below:



   (a) What is the Fourier Transform?
   Solution(recall that $\omega_0 = 2\pi f_0$ and $T_0 = \frac{1}{f_0}$):

$$D_n = \frac{1}{T_0}\int_0^{2a} e^{-j\omega_0 nt}dt$$

$$D_n = \frac{1}{T_0}\frac{e^{-j\omega_0 nt}}{-j\omega_0 n}\Big|_0^{2a}$$

$$D_n = \frac{1}{T_0}\frac{e^{-j\omega_0 n2a} - 1}{-j\omega_0 n}$$

$$D_n = \frac{1}{T_0}\frac{e^{-j\omega_0 n2a} - e^{j\omega_0 na}e^{-j\omega_0 na}}{-j\omega_0 n}$$

$$D_n = \frac{1}{T_0}\frac{(e^{-j\omega_0 na} - e^{j\omega_0 na})e^{-j\omega_0 na}}{-j\omega_0 n}$$

$$D_n = \frac{\sin(\omega_0 na)e^{-j\omega_0 na}}{\pi n}$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} \frac{\sin(\omega_0 na)e^{-j\omega_0 na}}{\pi n}\delta(\omega - n\omega_0)$$

(b) What is the duty cycle?
Solution: Duty cycle is the fraction of the cycle that the signal is on, $\frac{2a}{T_0}$

(c) What is the approximate bandwidth?
Solution: The bandwidth is the reciprocal of the pulse length, $BW = \frac{1}{2a}Hz$

3. If the signal given in problem 2 is shifted so that a pulse is centered on the origin, what happens to the:

(a) Magnitude spectrum?
Solution: NOTHING

(b) Phase spectrum?
Solution: It modified proportionally to the phase shift, which in this case corresponds to multiplication by $e^{j\omega_0 na}$ so the $e^{-j\omega_0 na}$ term goes away.

4. A baseband signal, $s(t)$ has a bandwidth of 1000 Hz. If $x(t) = s(t)\cos(2\pi 20,000t)$:

(a) Sketch the magnitude spectrum of $x(t)$. (Note: I don't care what you decide $s(t)$'s spectrum looks like, as long as it has the correct bandwidth)
Solution: any picture that you drew that was symmetric and has energy only from 19,000 to 21,000 Hz (and the same in negative frequencies) is OK. There were many pictures like this in lecture on 1/27/2011, but if you are unsure come to office hours or ask on Tuesday 2/1/2011!

(b) What is the bandwidth of $x(t)$?
Solution: 2000Hz

(c) Given that the power of $s(t)$ is A, what is the power of $x(t)$?
Solution: $\frac{A}{2}$

5. What is the Fourier Transform of $\cos(w_0 t) + \cos^2(\omega_0 t)$? (Hint, if you're integrating you're doing more work than you need to do!)
Solution: Recall

$$\cos(\omega_0 t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

so

$$\cos^2(\omega_0 t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

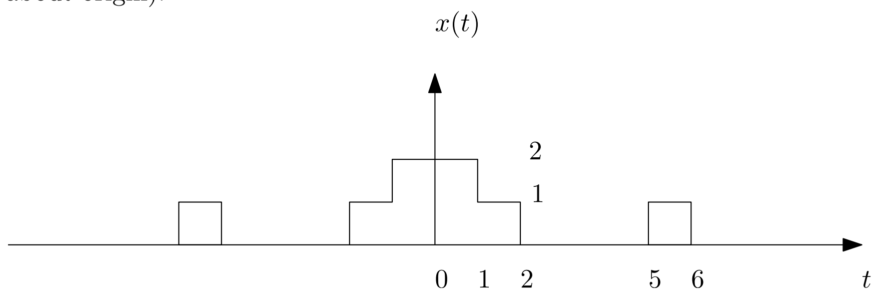
$$\cos^2(\omega_0 t) = \frac{e^{j2\omega_0 t} + 1 + 1 + e^{-j2\omega_0 t}}{4}$$

so the FT of the expression is:

$$\frac{1}{2}\delta(\omega - \omega_0) + \frac{1}{2}\delta(\omega + \omega_0) + \frac{1}{4}\delta(\omega - 2\omega_0) + \frac{1}{4}\delta(\omega + 2\omega_0) + \frac{1}{2}\delta(\omega)$$

6. Complete a and b:

(a) Use Fourier Transform properties to write the Fourier Transform of the below time signal(symmetric about origin):



Solution: We could write this signal as

$$x(t) = \Pi(\frac{t}{4}) + \Pi(\frac{t}{2}) + \Pi(\frac{t}{1}) * [\delta(t-5.5) + \delta(t+5.5)]$$

By the time shifting property $\delta(t-\tau) \leftrightarrow e^{-j2\pi\tau f}$so

$$X(f) = 4sinc(\pi 4f) + 2sinc(\pi 2f) + [e^{-j2\pi 5.5f} + e^{j2\pi 5.5f}]sinc(\pi f)$$

or

$$X(f) = 4sinc(\pi 4f) + 2sinc(\pi 2f) + 2\cos(2\pi 5.5f)sinc(\pi f)$$

notice that the cosine in the last term of the final equation has frequency as the independent variable.

(b) Sketch two other signals that would have the same exact magnitude spectrum.
Solution: Any two sketches with the signal shifted in time are OK.

7. A channel has the following transfer function:

$$H(f) = rect(\frac{f}{100} - 4.5) \cdot (160 - (\frac{f-500}{8})^2)e^{-j2\pi f(0.5)}$$

(a) What is the bandwidth of this channel?
Solution: The quadratic portion of the transfer function has infinite BW, the exponential at the end relates only to phase and since the rect spans 400 to 500 Hz in positive frequency, BW=100 Hz

(b) Does this channel have linear phase? If so, what is the delay through the channel?
Solution: Yes, because -$2\pi(0.5)f$ is a linear function of f. By the equation $t_d = -\frac{1}{2\pi}\frac{d\theta(f)}{df}$ the delay is .5 seconds.

(c) Is this channel linear?
Solution: Yes, only linear systems have meaningful transfer functions.

(d) Is this channel distortion free? If not, what is the source of the distortion?
Solution: No, because although it has linear phase it does not have constant gain across the passband.

(e) Plot or sketch the magnitude response of this channel.
Solution: The sketch should show a parabola that has roots at approximately 400 Hz and 600Hz gated by a rectangle function that goes from 400 to 500 Hz. The given equation was intended to result in a parabola with roots near 400 and 500 Hz, if your sketch looks like that because of something I told you in office hours, no points will be deducted.

8. Given that $x(t) = rect(t)$:

(a) Find the ESD via a product of Fourier Transforms.
Solution: $X(f) = sinc(\pi f)$ so $\Psi_x(f) = X(f)X^*(f) = X(f)X(f) = sinc^2(\pi f)$

(b) Find the autocorrelation function $\Psi_x(\tau)$.
Solution: $\Psi_x(\tau) = \Delta(\frac{t}{\tau}) = \Delta(\frac{t}{2})$

(c) Using $\Psi_x(\tau)$ verify the ESD you found in part a.
Solution: $\Delta(\frac{t}{2}) \leftrightarrow sinc^2(\pi f)$which is the same result as found above. Note: All these transform pairs are in the table on p 107.