MONODROMY GROUPS OF HURWITZ-TYPE PROBLEMS

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ABSTRACT. We solve the Hurwitz monodromy problem for degree 4 covers. That is, the Hurwitz space $\mathcal{H}_{4,g}$ of all simply branched covers of $\mathbb{P}^1$ of degree 4 and genus g is an unramified cover of the space $\mathcal{P}_{2g+6}$ of $(2g + 6)$-tuples of distinct points in $\mathbb{P}^1$. We determine the monodromy of $\pi_1(\mathcal{P}_{2g+6})$ on the points of the fiber. This turns out to be the same problem as the action of $\pi_1(\mathcal{P}_{2g+6})$ on a certain local system of $\mathbb{Z}/2$-vector spaces. We generalize our result by treating the analogous local system with $\mathbb{Z}/N$ coefficients, $3 \nmid N$, in place of $\mathbb{Z}/2$. This in turn allows us to answer a question of Ellenberg concerning families of Galois covers of $\mathbb{P}^1$ with deck group $(\mathbb{Z}/N)^2{:}S_3$.

A ramified cover C of $\mathbb{P}^1$ of degree d is said to have simple branching if the fiber over every branch point has $d-1$ distinct points. Another way to say this is that for each branch point p, the permutation of the sheets of the cover induced by a small loop around p is a transposition, i.e., a permutation of cycle-shape $21 \ldots 1$. An Euler characteristic argument (or the Hurwitz formula) shows that the number of branch points is $b := 2g + 2d - 2$, where g is the genus of C. Let $\mathcal{H}_{d,g}$ be the Hurwitz space, consisting of all such covers, up to isomorphism as covers. This is an irreducible smooth algebraic variety. There is an obvious map from $\mathcal{H}_{d,g}$ to the space $\mathcal{P}_b$ of unordered b-tuples of distinct points in $\mathbb{P}^1$. This is an unramified cover, so it induces a homomorphism from $G := \pi_1(\mathcal{P}_b)$ to the symmetric group on the points of a fiber. We determine the image in the case $d = 4$; this answers this case of a question posed explicitly in [9] and implicit in earlier work. We call this image G_2; the subscript reflects that this turns out to be the case $N = 2$ of a more general construction considered below.

Our formulation of the problem reflects its topological nature, but usually one thinks of $\mathcal{H}_{d,g}$ and $\mathcal{P}_b$ as irreducible algebraic varieties, so that the function field of $\mathcal{H}_{d,g}$ is a finite extension of that of $\mathcal{P}_b$. Then

G_2 is the Galois group of the associated Galois extension. Even the degree of this extension was unknown.

Theorem 1. *Let $g > 1$. Then monodromy group G_2 of $\mathcal{H}_{4,g} \to \mathcal{P}_{b=2g+6}$ fits into the split exact sequence*

$$(1) \qquad 1 \to \prod_\Omega \mathrm{Sp}(2g+2, \mathbb{Z}/2) \to G_2 \to \mathrm{PSp}(2g+4, \mathbb{Z}/3) \to 1,$$

where $\Omega = \mathbb{P}^{2g+3}(\mathbb{Z}/3)$ and $\mathrm{PSp}(2g+4, \mathbb{Z}/3)$ permutes the factors of the product in the obvious way.

Remark. The $g = 0, 1$ cases are exceptional. If $g = 0$ then the left term of (1) should be $3^{40}{:}2^{16}$ instead of S_3^{40}, and the sequence is nonsplit. If $g = 1$ then the left term should be $A_6^{364}{:}2^{168}$ rather than S_6^{364}, and we did not determine whether the sequence splits. (We use ATLAS notation for group structures [8].)

The fact that G_2 lies in a group fitting into an exact sequence like (1) is due to Eisenbud, Elkies, Harris and Speiser [9]; see also [7] and [15]. So our result says that G_2 is as large as possible. In section 1 we will review what we need from [9] and then prove the theorem.

In section 2 we treat two generalizations of this that are similar to each other. The degree-4 Hurwitz monodromy problem is very closely related to a certain local system of $\mathbb{Z}/2$-vector spaces over $\mathcal{P}_b$. Namely, $\mathcal{H}_{3,g+1}$ is also an unramified cover of $\mathcal{P}_b$, and over $\mathcal{H}_{3,g+1}$ there is a universal family $\mathcal{C}_{3,g+1}$ of simply branched 3-fold covers of $\mathbb{P}^1$. (Existence of this family is not hard to see, and is proven in great generality in [11].) We write π for the composition $\mathcal{C}_{3,g+1} \to \mathcal{H}_{3,g+1} \to \mathcal{P}_b$. If $N \geq 0$, then we consider the sheaf $\mathcal{V}_N := R^1\pi_*(\mathbb{Z}/N)$ on $\mathcal{P}_b$, which we recall is the sheaf associated to the presheaf $U \mapsto H^1(\pi^{-1}(U); \mathbb{Z}/N)$; the case $N = 0$ corresponds to $\mathbb{Z}$ coefficients. $\mathcal{V}_N$ is a local system of $\mathbb{Z}/N$-modules equipped with symplectic forms; the fiber over a point $p = (p_1, \ldots, p_b) \in \mathcal{P}_b$ is $H^1(\pi^{-1}(p), \mathbb{Z}/N)$, which is the direct sum of the $H^1(C; \mathbb{Z}/N)$ as C varies over the points of $\mathcal{H}_{3,g+1}$ lying above p. As we explain in section 1, the monodromy of $\pi_1(\mathcal{P}_b)$ on $\mathcal{V}_2$ is exactly the Hurwitz monodromy group in degree 4, which we called G_2. So we define G_N as the monodromy group on $\mathcal{V}_N$. We have completely determined G_N when $3 \nmid N$, except for the cases $g = 0$ or 1 and the question of whether the exact sequence (2) below splits.

Theorem 2. *Suppose $3 \nmid N$ and $g \geq 0$ ($g > 1$ if N is even). Then the monodromy group G_N of $\mathcal{V}_N$ fits into an exact sequence*

$$(2) \qquad 1 \to \prod_\Omega \mathrm{Sp}(2g+2, \mathbb{Z}/N) \to G_N \to \mathrm{PSp}(2g+4, \mathbb{Z}/3) \to 1,$$

where Ω and the action of $\mathrm{PSp}(2g+4, \mathbb{Z}/3)$ are as in theorem 1.

Question. *What happens if $3|N$?* The most extreme case is G_0, the case of integer coefficients, which determines G_N for all N. The congruence subgroup property of $\mathrm{Sp}(2g, \mathbb{Z})$ probably reduces this to the determination of G_{3^n} for all n. But the congruence subgroup property requires $g > 1$, so it would only apply for $b \geq 8$.

Finally, we use theorem 2 to answer a question of Ellenberg [10], which we motivate by reinterpreting the Hurwitz monodromy problem. If $C \to \mathbb{P}^1$ is connected and simply branched of degree 4, then its associated Galois cover has deck group S_4. The Hurwitz monodromy can be regarded as the action of $\pi_1(\mathcal{P}_b)$ on the family of all Galois covers of $\mathbb{P}^1$ that have deck group S_4 and satisfy a condition which is a rephrasing of the simple branching of $C \to \mathbb{P}^1$. What makes the degree-4 case special is that S_4 is solvable: it is a semidirect product $2^2{:}S_3$. Ellenberg essentially asked: what happens when the 2^2 is replaced by the elementary abelian group p^2 for some prime $p > 3$? We show that the resulting monodromy group fits into a split exact sequence like (1), with $\mathbb{Z}/2$ replaced by $\mathbb{Z}/p$.

Here is a precise formulation of his question, in a more general context. Let X_N be the semidirect product $N^2{:}S_3$, with S_3 acting by permuting triples of elements of $\mathbb{Z}/N\mathbb{Z}$ with sum 0. Consider Galois covers of $\mathbb{P}^1$ with Galois group X_N and b branch points, such that the small loops around them correspond to involutions in X_N. When N is even we require further that these involutions have nontrivial image in S_3. Let $\mathcal{E}_N$ be the set of isomorphism classes of such covers; this is a local system of finite sets over $\mathcal{P}_b$, and Ellenberg's question can be phrased: what is the image $\bar{G}_N$ of the monodromy action of $G = \pi_1(\mathcal{P}_b)$ on a fiber of $\mathcal{E}_N$? This type of problem was considered by Biggers and Fried [5], who showed that $\bar{G}_N$ is transitive on the fiber, so $\mathcal{E}_N$ is irreducible. We can go further: for N prime to 3, we have completely determined the structure of $\bar{G}_N$, except for $b = 4$ or 6 when N is even. Theorem 2 fairly easily implies the following theorem:

Theorem 3. *Suppose $3 \nmid N$ and $b > 4$ ($b > 8$ if N is even). Then the monodromy group $\bar{G}_N$ of $\mathcal{E}_N \to \mathcal{P}_b$ fits into the split exact sequence*

$$(3) \qquad 1 \to \prod_{\Omega} \mathrm{PSp}(b-4, \mathbb{Z}/N) \to \bar{G}_N \to \mathrm{PSp}(b-2, \mathbb{Z}/3) \to 1,$$

where $\Omega = \mathbb{P}^{b-3}(\mathbb{Z}/3)$ and $\mathrm{PSp}(b-2, \mathbb{Z}/3)$ permutes the factors of the product in the obvious way.

Remarks. The expressions $\mathrm{Sp}(\dots)$ make sense because b always turns out to be even. Also, by $\mathrm{PSp}(b-4, \mathbb{Z}/N)$ we mean the quotient of $\mathrm{Sp}(b-4, \mathbb{Z}/N)$ by its center, which is an elementary abelian 2-group.

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1. Proof of theorem 1

In this section we will review the relevant results of [9] and then prove theorem 1. The key feature of the $d = 4$ case of the Hurwitz monodromy problem is that S_4 is solvable, so that a degree 4 cover $C \to \mathbb{P}^1$ determines a number of related covers of $\mathbb{P}^1$, shown in Figure 1. To organize them we will use subscripts to indicate their degrees over $\mathbb{P}^1$. If $C \to \mathbb{P}^1$ is connected and simply branched of degree 4, with b branch points $p_1, \dots, p_b$, then there is an associated surjection $\pi_1(C - \{p_i\}) \to S_4$, well-defined up to conjugacy by an element of S_4, sending small loops around the p_i to transpositions. The corresponding Galois cover C_{24} has S_4 as its deck group, and we define $C_6 := C_{24}/V_4$ and $C_2 := C_{24}/A_4$, where V_4 is Klein's Viergruppe. C itself is C_{24}/S_3 for one of the four conjugate S_3's in S_4, so we could write C_4 for C. We will refer to the covers $C_{24}/D_8 \to \mathbb{P}^1$, for the three conjugate D_8's in S_4, as "the 3 C_3's". As explained in [9, sec. 4], C_2 is hyperelliptic, $C_2 \to \mathbb{P}^1$ has simple branching over the p_i, and $C_{24} \to C_6$ and $C_6 \to C_2$ are unramified with deck groups 2^2 and 3. The genera of C_6 and C_2 are $3g + 4$ and $g + 2$. Each C_3 is simply branched over $\mathbb{P}^1$, with b branch points and genus $g + 1$. These data can be obtained with the Hurwitz formula or by topological picture-drawing like that in Figure 2.

The interplay between these covers allows one to describe the fiber of $\mathcal{H}_{4,g} \to \mathcal{P}_b$ concretely. Each of the C_3's represents the same point of $\mathcal{H}_{3,g+1}$, and C_2 represents a point of $\mathcal{H}_{2,g+2}$, yielding a factorization of $\mathcal{H}_{4,g} \to \mathcal{P}_b$ as $\mathcal{H}_{4,g} \to \mathcal{H}_{3,g+1} \to \mathcal{H}_{2,g+2} = \mathcal{P}_b$. It is usually more convenient to work with Galois covers, so we remark that $C_4, C_4' \in \mathcal{H}_{4,g}$ are equivalent as covers (i.e., are the same point of $\mathcal{H}_{4,g}$) if and only if the Galois covers C_{24} and C_{24}' are. This follows from the conjugacy of index-4 subgroups of S_4. The same argument shows that $C_3, C_3' \in \mathcal{H}_{3,g+1}$ are equivalent if and only if the Galois covers C_6, C_6' are. Because
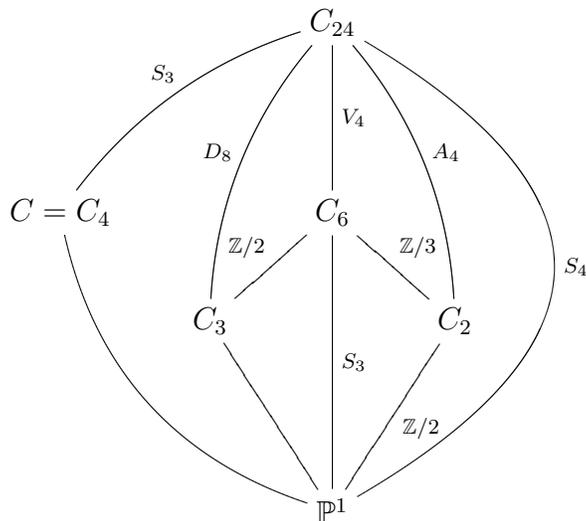
FIGURE 1. Covers associated to a degree 4 cover $C \to \mathbb{P}^1$.

of this, we will sometimes refer to (say) C_6 in order to specify a point of $\mathcal{H}_{3,g+1}$.

Now we describe the fibers of $\mathcal{H}_{2,g+2}$, $\mathcal{H}_{3,g+1}$ and $\mathcal{H}_{4,g}$ over a b-tuple $(p_1, \dots, p_b) \in \mathcal{P}_b$ in terms of the possibilities for the Galois covers C_2, C_6 and C_{24}. There is only one C_2 with specified branch points $p_1, \dots, p_b$. The unramified $\mathbb{Z}/3$-covers of C_2 that are Galois over $\mathbb{P}^1$ are in bijection with the hyperplanes h in $H_1(C_2; \mathbb{Z}/3)$ that are preserved by the hyperelliptic involution α of C_2. The condition that the Galois group be S_3 rather than $\mathbb{Z}/6$ is that α act on $H_1(C_2; \mathbb{Z}/3)/h$ by negation. Since α acts by negation on all of $H_1(C_2; \mathbb{Z}/3)$, these conditions on h are vacuous, and the possibilities for C_6 are in bijection with $\mathbb{P}H^1(C_2; \mathbb{Z}/3)$.

Once $C_6 \to \mathbb{P}^1$ is fixed, the possibilities for C_{24} are parameterized in a similar but more complicated way. The unramified covers of C_6 with deck group 2^2 that are Galois over $\mathbb{P}^1$ are in bijection with the codimension-two subspaces L of $H_1(C_6; \mathbb{Z}/2)$ which are preserved by $S_3 = \mathrm{Gal}(C_6/\mathbb{P}^1)$. And the condition for the Galois group to be S_4 rather than some other extension $2^2.S_3$ is that S_3 acts on $H_1(C_6; \mathbb{Z}/2)/L$ in the same way that $S_3 = S_4/V_4$ acts on V_4. Dualizing, the choices for C_{24} correspond to the subgroups $(\mathbb{Z}/2)^2$ of $H^1(C_6; \mathbb{Z}/2)$ which S_3 preserves and acts on by its 2-dimensional irreducible representation, which permutes triples of elements of $\mathbb{Z}/2$ with sum 0. We write $\mathbb{P}V(C_6)$

for this set of subspaces, the notation reflecting the fact that it is a projective space in a non-obvious way.

To see this, fix one of the three C_3's, and regard $H^1(C_3; \mathbb{Z}/2)$ as embedded in $H^1(C_6; \mathbb{Z}/2)$ under pullback. Every one of the 2-dimensional subspaces of $H^1(C_6; \mathbb{Z}/2)$ considered above contains a unique $\mathbb{Z}/2$ lying in $H^1(C_3; \mathbb{Z}/2)$, and every $\mathbb{Z}/2$ in $H^1(C_3; \mathbb{Z}/2)$ lies in a unique one of these 2-dimensional subspaces. So $\mathbb{P}V(C_6)$ is in bijection with $\mathbb{P}H^1(C_3; \mathbb{Z}/2)$. The three C_3's all give the same projective space structure, so the choices for C_{24}, given C_6, correspond to points of $\mathbb{P}V(C_6) \cong \mathbb{P}^{2g+1}(\mathbb{Z}/2)$. We can even be a little fancier and define $V(C_6)$ as the union of the three $H^1(C_3; \mathbb{Z}/2)$'s, modulo identification under the group $\mathbb{Z}/3$ of deck transformations. Then $\mathbb{P}V(C_6)$ is indeed the projectivization of $V(C_6)$.

In summary, once $p_1, \ldots, p_b$ are fixed, the possibilities for $C = C_4$ are in bijection with the ordered pairs (C_6, C_{24}), where C_6 corresponds to an element of $\mathbb{P}H^1(C_2; \mathbb{Z}/3)$ and C_{24} to an element of $\mathbb{P}V(C_6)$. All of these constructions can be carried out simultaneously for all b-tuples (this is the basic property of Hurwitz spaces). The result is that $\mathcal{H}_{4,g}$ is an unramified cover of $\mathcal{P}_b$, which factors as $\mathcal{H}_{4,g} \to \mathcal{H}_{3,g+1} \to \mathcal{P}_b$, with a fiber of the second map parameterizing the possible choices for C_6 (or equivalently C_3). The fiber of the first map over a chosen C_6 is $\mathbb{P}V(C_6) \cong \mathbb{P}^{2g+1}(\mathbb{Z}/2)$, parameterizing the possible choices for C_{24}, given C_6. So a fiber of $\mathcal{H}_{4,g}$ over $\mathcal{P}_b$ consists of $|\mathbb{P}^{2g+3}(\mathbb{Z}/3)|$ many copies of $\mathbb{P}^{2g+1}(\mathbb{Z}/2)$.

We are interested in the monodromy action of $G := \pi_1(\mathcal{P}_b)$ on this fiber. It obviously respects the symplectic structure on $H^1(C_2; \mathbb{Z}/3)$, and the stabilizer of C_6 preserves the symplectic structure on $V(C_6) = H^1(C_3; \mathbb{Z}/2)$. Therefore the image G_2 can be no larger than in (1).

Having reviewed the results of [9], we will now prove the theorem. We will write $\beta_1, \ldots, \beta_{b-1}$ for the standard generators for the spherical braid group on b strands, which is G.

Lemma 4. *The monodromy action of any β_i on a fiber $\mathbb{P}H^1(C_2; \mathbb{Z}/3)$ of $\mathcal{H}_{3,g+1} \to \mathcal{P}_b$ is a symplectic transvection, and G acts by the full projective symplectic group* $\mathrm{PSp}(2g + 4, \mathbb{Z}/3)$.

Proof. This is due to Cohen [7]; the key point is the following. Let L be a simple loop in $\mathbb{P}^1$ encircling p_i and p_{i+1} but none of the other branch points. Then L lifts to a closed loop $\tilde{L}$ on C_2. The monodromy of β_i on C_2 is a Dehn twist in $\tilde{L}$. (For a visual proof see figs. 5–7 of [4, ch. 1] and the surrounding text.) This acts on cohomology by a transvection.

For the second statement we appeal to Clebsch's theorem [6, pp. 224–225] that G is transitive on the sheets of $\mathcal{H}_{3,g+1} \to \mathcal{P}_b$, which is to
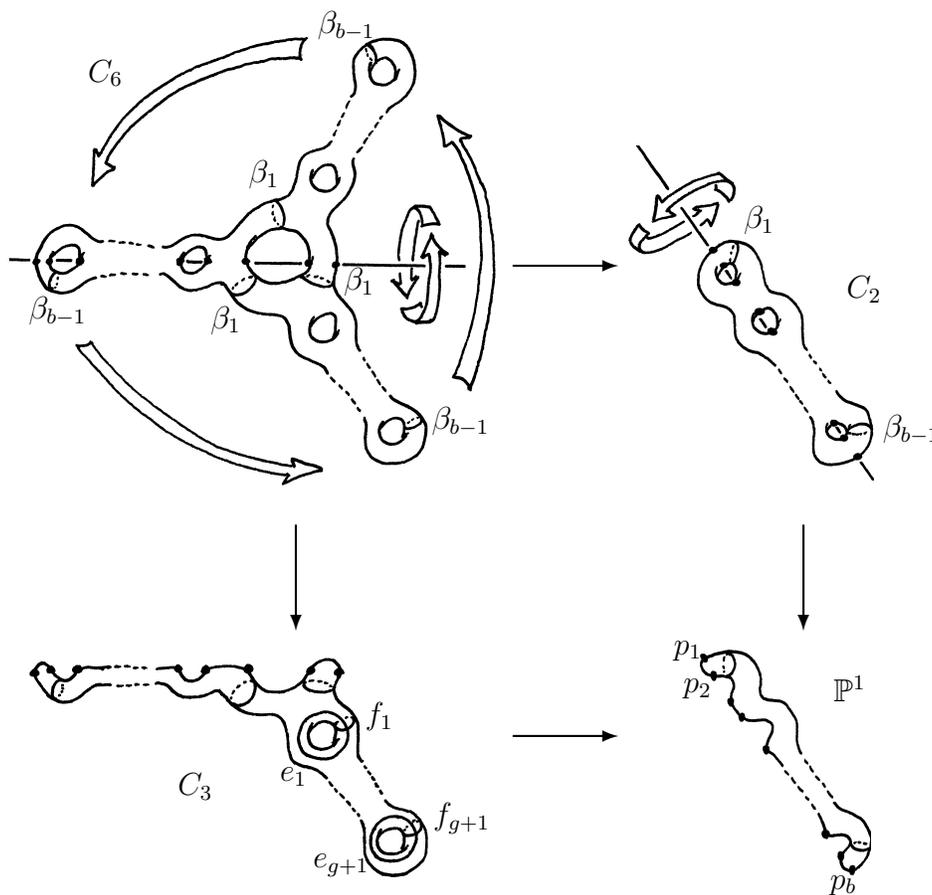
FIGURE 2. Covers of $\mathbb{P}^1$ associated to $C \to \mathbb{P}^1$.

say that it is transitive on $\mathbb{P}H^1(C_2; \mathbb{Z}/3)$. The G-conjugates of the β_i therefore give all the transvections, which are well-known to generate the symplectic group. $\qquad \square$

Now pick a point of $\mathcal{H}_{3,g+1}$; this corresponds to a cover C_6 (equivalently, C_3) and also to an element of $\mathbb{P}H^1(C_2; \mathbb{Z}/3)$, say the one in which β_1 acts by a transvection. We will abbreviate $V(C_6)$ to V. Now we consider the subgroup H of G whose monodromy sends C_6 to itself, and the action of H on the fiber $\mathbb{P}V$ of $\mathcal{H}_{4,g}$ over C_6.

Lemma 5. *H contains β_1, which acts trivially on V, and $\beta_3, \ldots, \beta_{b-1}$, which act by symplectic transvections. And H acts on V by the full projective symplectic group $\mathrm{PSp}(V) \cong \mathrm{Sp}(2g+2, \mathbb{Z}/2)$.*

Proof. Before beginning the proof proper we make V concrete. Figure 2 shows the maps $C_6 \to C_3 \to \mathbb{P}^1$ and $C_6 \to C_2 \to \mathbb{P}^1$. The picture of

$\mathbb{P}^1$ shows the branch points $p_1, \ldots, p_b$. The loop encircling p_1 and p_2 has a lift to C_2, marked β_1. We use this notation because β_1 acts on C_2 as a Dehn twist in that loop, which was called $\tilde{L}$ in the proof of lemma 4. Now, C_6 is defined as the cover of C_2 corresponding to the elements of $\pi_1(C_2)$ having trivial intersection (mod 3) with $\tilde{L}$, and is shown. The deck group acts by the obvious $\mathbb{Z}/3$ rotation. Next, there are 3 involutions in $S_3 = \mathrm{Gal}(C_6/\mathbb{P}^1)$, one of which is the $\mathbb{Z}/2$ rotation around the horizontal axis. The quotient C_3 is shown, together with the branch points of $C_6 \to C_3$ and a basis $e_1, f_1, \ldots, e_{g+1}, f_{g+1}$ of $H^1(C_3)$. If we indicate lifts of these loops to the 3 'arms' of C_6 by $e_j^{(i)}$ and $f_j^{(i)}$, for $i = 0, 1, 2$ and $j = 1, \ldots, g+1$, then up to relabeling, the pullback V^{01} of $H^1(C_3)$ is spanned by the $e_j^{(0)} + e_j^{(1)}$ and $f_j^{(0)} + f_j^{(1)}$. The other two C_3's give the same result but with different superscripts. The space V is the union of these three vector spaces, modulo cyclic permutation of the upper labels 0, 1 and 2.

Now, β_1 lies in H, so it lifts to C_6; "the" action on C_6 is only well-defined up to composition with deck transformations. But these act trivially on V, by the definition of V, so the action of β_1 on V may be computed from any one of the three lifts of β_1. One of these lifts is the composition of the Dehn twists in the three loops marked β_1. This obviously leaves the $e_j^{(i)}$ and $f_j^{(i)}$ unperturbed, so β_1 acts trivially on V.

The same analysis applies to β_{b-1}, one of whose lifts to C_6 is the composition of the three indicated Dehn twists. Its restriction to V^{01} is the transvection in $f_{g+1}^{(0)} + f_{g+1}^{(1)}$ (with respect to the symplectic form pulled back from C_3, not the one on $H^1(C_6)$). This proves that β_{b-1} acts on V as a transvection. The argument is the same for $\beta_3, \ldots, \beta_{b-2}$.

We remark that up to this point, the argument works perfectly well with $\mathbb{Z}$ coefficients in place of $\mathbb{Z}/2$.

Finally, we again use Clebsch's transitivity theorem, this time applied to the fibers of $\mathcal{H}_{4,g} \to \mathcal{P}_b$, to deduce that H acts transitively on the fiber of $\mathcal{H}_{4,g}$ over the point of $\mathcal{H}_{3,g+1}$ corresponding to C_6. This fiber is $\mathbb{P}V$. Since the image of H contains a transvection and is transitive on $\mathbb{P}V$, it contains all transvections, hence equals $\mathrm{PSp}(V)$. $\qquad\square$

Now we will consider the kernel K of $G \to \mathrm{PSp}(2g + 4, \mathbb{Z}/3)$ and its image K_2 in G_2, which is a subgroup of the direct product appearing in (1). We will improve the previous lemma by showing that K has the same surjectivity properties we just established for H; then we will show that this is a fierce restriction on K_2.

Lemma 6. *The projection of K_2 to any factor of $\prod_\Omega \mathrm{PSp}(2g+2, \mathbb{Z}/2)$ is surjective.*

Proof. Because G permutes the factors transitively, it suffices to treat any one, say $\mathrm{PSp}(V)$. Now, K is normal in H, and H surjects to $\mathrm{PSp}(V)$, so the image of K is a normal subgroup of $\mathrm{PSp}(V)$. It also contains the transvection β_{b-1}^3. Therefore it contains all transvections, hence all of $\mathrm{PSp}(V)$. $\qquad\square$

If S is a group, then we call a subgroup of a product of copies of S diagonally embedded if it projects isomorphically to each factor. The language expresses the fact that it is *the* diagonal subgroup, up to automorphisms of the factors.

Lemma 7. *Let S be a nonabelian simple group, Ω a finite set, and K_2 a subgroup of $\prod_\Omega S$ that surjects to each factor. Then $K_2 \cong S^n$ for some n, and there is a partition $\Omega = \Omega_1 \coprod \cdots \coprod \Omega_n$, such that the ith factor of K_2 is diagonally embedded in $\prod_{\Omega_i} S$, for each i.*

Proof. We first remark that a product of copies of a nonabelian simple group is a product in only one way, since the factors are the normal simple subgroups. We will also use the following standard fact [17, ch. 2, thm. 4.19]: if A, A' are groups, then the subgroups J of $A \times A'$ are in bijection with the 5-tuples (B, B', C, C', ϕ) where B and B' are subgroups of A and A', C and C' are normal subgroups of B and B', and ϕ is an isomorphism $B/C \cong B'/C'$. (B and B' are the projections of J to A and A', C and C' are the intersections of J with the factors, and J is the preimage of the graph of ϕ under $B \times B' \to B/C \times B'/C'$.)

The proof is by induction on $|\Omega|$, the case of a singleton being trivial. So suppose $|\Omega| > 1$, choose a point $\omega \in \Omega$, and define $\Omega' := \Omega - \{\omega\}$. We apply the above with $A = \prod_{\{\omega\}} S \cong S$, $A' = \prod_{\Omega'} S$ and $J = K_2 \subseteq A \times A'$. By the assumed surjectivity, B surjects to A, and B' surjects to each factor of $\prod_{\Omega'} S$. By induction, $B' \cong S^m$ for some m, and there is a partition $\Omega' = \Omega'_1 \coprod \cdots \coprod \Omega'_m$ such that the ith factor of B' is diagonally embedded in $\prod_{\Omega'_i} S$. Now, because $B \cong S$ is simple, C is either all of B or is trivial. In the first case, $B'/C' \cong B/C = 1$, so $C' = B'$ also. Then $K_2 = B \times B' \cong S^{m+1}$, with its ith factor diagonally embedded in $\prod_{\Omega_i} S$, where $\Omega_1 = \Omega'_1, \ldots, \Omega_m = \Omega'_m$, $\Omega_{m+1} = \{\omega\}$.

In the second case, $B'/C' \cong B/C \cong S$, so $K_2 \subseteq B' \times B = S^m \times S$ is the graph of a surjection $B' \to B$. Because S is nonabelian simple, any normal subgroup of S^m is the product of some of the given factors. Therefore the kernel of $B' \to B$ consists of $m-1$ factors of S^m, say all but the first. We conclude that $K_2 \subseteq B' \times B$ is generated by a diagonally embedded copy of S in each of $\prod_{\Omega'_2} S, \ldots, \prod_{\Omega'_m} S$, together

with the graph of an isomorphism from a diagonally embedded copy of S in $\prod_{\Omega_1'} S$ to $B = \prod_{\{\omega\}} S \cong S$. It follows that $K_2 \cong S^m$, with its ith factor diagonally embedded in $\prod_{\Omega_i} S$, where $\Omega_1 = \Omega_1' \cup \{\omega\}$ and $\Omega_2 = \Omega_2', \ldots, \Omega_m = \Omega_m'$. $\qquad\square$

Proof of theorem 1: We will write S for $\mathrm{PSp}(2g+2, \mathbb{Z}/2)$. We know by lemma 4 that G_2 surjects to $\mathrm{PSp}(2g+4, \mathbb{Z}/3)$, so to establish the exact sequence it suffices to show that K_2 is the full direct product $\prod_\Omega S$. Since $g > 1$, S is simple. It follows from lemmas 6 and 7 that there is a partition $\Omega = \Omega_1 \coprod \cdots \coprod \Omega_n$ such that $K_2 \cong S^n$, its ith factor being diagonally embedded in $\prod_{\Omega_i} S$. Now, G's action on K_2 permutes the factors of K_2, in a manner compatible with G's action on Ω. Therefore G respects the partition. But $\mathrm{PSp}(2g+4, \mathbb{Z}/3)$ acts primitively on Ω, so either all the Ω_i are singletons or else there is only one Ω_i. In the first case we have $K_2 = \prod_\Omega S$, as desired. So we must rule out the case where K_2 is isomorphic to S and is diagonally embedded in $\prod_\Omega S$. We will do this by exhibiting a nontrivial element of K_2 with trivial projection to one factor. By lemma 5, β_1^3 acts trivially on V. On the other hand, β_1^3 is G-conjugate to β_{b-1}^3, whose image in $\mathrm{PSp}(V)$ is nontrivial, by the same lemma.

Finally, we show that the sequence (1) splits. Because K_2 has no center, a standard result [17, ch. 2, thm. 7.11] shows that the structure of G_2 is determined by the homomorphism $G_2/K_2 \to \mathrm{Out}(K_2)$. Since S is a nonabelian simple group with trivial outer automorphism group, $\mathrm{Out}(K_2) = \mathrm{Sym}(\Omega)$. Also, the homomorphism $\mathrm{PSp}(2g+4, \mathbb{Z}/3) \to \mathrm{Sym}(\Omega)$ is the permutation action on Ω. Since there is exists a split extension giving this homomorphism, and the homomorphism determines G_2, G_2 must split. $\qquad\square$

In the cases $g = 0, 1$, lemma 7 no longer applies because the groups $\mathrm{PSp}(2, \mathbb{Z}/2) \cong S_3$ and $\mathrm{PSp}(4, \mathbb{Z}/2) \cong S_6$ are not simple; they are extensions of $\mathbb{Z}/2$ by the simple group $S' = [S, S]$. One can describe the permutation representation of $\pi_1(\mathcal{P}_b)$ on the fiber of $\mathcal{H}_{4,g} \to \mathcal{P}_b$ in a manner suitable for computer calculation, and for $g = 0$ we discovered $|G_2| = 3^{40}.2^{16}|\mathrm{PSp}(4, \mathbb{Z}/3)|$, so $K_2 = 3^{40}.2^{16} \subseteq S_3^{40}$. For $g = 1$ the calculation exceeded our available computing power, so we proceeded as follows. An argument as in the proof of theorem 1 shows that $K_2' := K_2 \cap \prod_\Omega S'$ is either the full direct product $\prod_\Omega S'$ or is isomorphic to S' and is diagonally embedded in $\prod_\Omega S'$. (K_2' turns out to be the commutator subgroup of K_2, justifying the notation. This also holds in the $g = 0$ case.) A computer-aided calculation shows that K_2' is the full direct product $\prod_\Omega S'$. The crucial step is an analogue

of lemma 6 for K_2'. Namely, while β_i^3 lies in K_2, it does not lie in K_2' because transvections lie outside S'. Nonetheless, an explicit calculation shows that $[\beta_1^3, \beta_2^3]$ is a non-trivial element of K_2' which projects trivially to at least one factor S', hence $K_2' = \prod_\Omega S'$.

As described below, we wrote down an explicit faithful permutation representation of G_2/K_2', which was within reach of computer calculation. We found that G_2/K_2' is $2^{16}.\mathrm{PSp}(4, \mathbb{Z}/3)$ for $g = 0$ and $2^{168}.\mathrm{PSp}(6, \mathbb{Z}/3)$ for $g = 1$. Although we already knew this when $g = 0$, in this representation we could show that extension is not split, which was out of reach before killing K_2'. We did not apply sufficient computing power to determine whether or not it splits for $g = 1$. We carried out our computer calculations using GAP [12].

To describe our representation of G_2/K_2' we recall from [9, Section 1] the (faithful) permutation representation of G_2 on the collection Σ of S_4-orbits of b-tuples $(\sigma_1, \ldots, \sigma_b)$ of 2-cycles in S_4 such that $\sigma_1 \cdots \sigma_b = 1$ and $\langle \sigma_1, \ldots, \sigma_b \rangle = S_4$. Here, β_i acts by replacing σ_i by σ_{i+1} and σ_{i+1} by $\sigma_{i+1}^{-1}\sigma_i\sigma_{i+1}$ and leaving all other σ_j invariant, and S_4 acts by simultaneous conjugation on all elements of a b-tuple.

In a similar fashion we may identify Ω with the S_3-orbits of b-tuples of 2-cycles in S_3 so that if we fix an isomorphism $S_3 \cong S_4/V_4$, then the induced map $\Sigma \to \Omega$ is G_2-equivariant. If we fix $\omega \in \Omega$ to be the point corresponding to C_6 and write Σ_ω for the fiber over ω, then we may identify Σ_ω with $\mathbb{P}V$ and $S = \mathrm{PSp}(V)$ with the factor of $\prod_\Omega S$ over ω.

If we write H_2 for the stabilizer in G_2 of ω, then the representation $G_2 \to \mathrm{Sym}(\Omega)$ is equivalent to the left representation of G_2 on G_2/H_2. Moreover, if we write H_2^* for the kernel of the composite map $H_2 \to S \to \mathbb{Z}/2$, then K_2' is the intersection of all G_2-conjugates of H_2^* and hence is the kernel of the left representation of G_2 on $\Omega' = G_2/H_2^*$. In particular, given a set of coset representatives of G_2/H_2^* and a black box for identifying when two elements of G_2 lie in the same coset, it is easy to compute the representation $G_2 \to \mathrm{Sym}(\Omega')$: β_i takes the coset $\alpha_j H_2^*$ to the coset $\beta_i \alpha_j H_2^*$.

To construct representatives one takes a known subset $\alpha_1, \ldots, \alpha_m$, computes $\beta_i \alpha_j H_2^*$ for $i = 1, \ldots, b-1$ and $j = 1, \ldots, m$, adds any new cosets which arise to the known subset, and repeats until no new cosets are constructed.

To construct the black box observe that the elements γ_1, γ_2 represent the same coset if and only if $\gamma = \gamma_1^{-1}\gamma_2$ lies in H_2^*, and the latter occurs if and only if γ both stabilizes ω and lies in the kernel of $H_2 \to \mathbb{Z}/2$. For $g = 1$, the parity map $S \simeq S_6 \to \mathbb{Z}/2$ is not the restriction of the parity map $\mathrm{Sym}(\Sigma_\omega) \to \mathbb{Z}/2$, hence a little work is required to determine the

former; a transvection in S fixes $2^3 - 1$ lines and permutes the other $2^4 - 2^3$ in pairs, hence is a product of four 2-cycles in $\mathrm{Sym}(\Omega_\omega)$. If we chose ω to correspond to the b-tuple $(\sigma_1, \ldots, \sigma_b)$ with $\sigma_1 = \sigma_2 = (12)$ and $\sigma_3 = \cdots = \sigma_b = (23)$, then one can easily verify that $\beta_3, \ldots, \beta_{b-1}$ stabilize ω, they each act non-trivially on V, and they generate S. Since they generate S and are conjugate, they must all map to the nontrivial element of $\mathbb{Z}/2$ and the image of $s \in S$ under $S \to \mathbb{Z}/2$ is the parity of the length of s as a product in $\beta_3, \ldots, \beta_{b-1}$.

2. Proof of theorems 2 and 3

We first introduce a little notation for talking about $\mathcal{V}_N$. Choosing a point of $\mathcal{H}_{3,g+1}$ means choosing a simply branched cover $C_3 \to \mathbb{P}^1$, or equivalently the associated Galois cover $C_6 \to \mathbb{P}^1$. We define $V_N(C_6)$ to be $H^1(C_3; \mathbb{Z}/N)$, or more intrinsically as the union of the pullbacks to $H^1(C_6; \mathbb{Z}/N)$ of the three $H^1(C_3; \mathbb{Z}/N)$'s, modulo identifications by the action of $\mathbb{Z}/3$. For fixed $(p_1, \ldots, p_b) \in \mathcal{P}_b$, the fiber of $\mathcal{V}_N$ is $\oplus_{C_3} V_N(C_6)$, where the sum extends over the points $C_3 \in \mathcal{H}_{3,g+1}$ lying above $(p_1, \ldots, p_b)$. When $N = 2$, $V_2(C_6)$ is just $V(C_6)$ from section 1, giving the relation to the Hurwitz monodromy problem.

Now we can discuss the monodromy. The map $G \to \mathrm{PSp}(2g+4, \mathbb{Z}/3)$ is the same as in the previous section, corresponding to the action on $\Omega = \mathbb{P}H^1(C_2; \mathbb{Z}/3)$. As before, we write K for the kernel, which acts on $\mathcal{V}_N$ by a subgroup of $P_N := \prod_\Omega \mathrm{Sp}(2g + 2, \mathbb{Z}/N)$. Also, we saw in lemma 4 that β_1 acts on $H^1(C_2; \mathbb{Z}/3)$ as a transvection, so it distinguishes an element of Ω. We write H for the G-stabilizer of this point, C_6 for the corresponding S_3-cover of $\mathbb{P}^1$, and V_N for $V_N(C_6) \cong (\mathbb{Z}/N)^{2g+2}$.

Lemma 8. *If $g \geq 0$ and $N \geq 0$, then H acts on V_N as $\mathrm{Sp}(V_N)$.*

Proof. It suffices to prove this in the case $N = 0$, i.e., with $\mathbb{Z}$ coefficients. A'Campo [2, Thm. 1(2)] studied a particular representation of the braid group $B_{\mu+1}$, μ even, into $\mathrm{Sp}(\mu, \mathbb{Z})$. We have a representation of $B_{b-2} = \langle \beta_3, \ldots, \beta_{b-1} \rangle \subseteq H$ into $\mathrm{Sp}(2g + 2, \mathbb{Z})$. (Recall that $b = 2g + 6$.) In both cases the braid generators act by transvections in primitive lattice vectors (this uses lemma 5, whose proof goes through perfectly well over $\mathbb{Z}$). These representations are essentially unique, since the transvections in two nonproportional vectors braid if and only if pairing the vectors yields ± 1. Therefore our representation contains his, with $\mu = 2g + 2$, and we even have an extra generator. He proves that the image of his representation contains the level-2 congruence subgroup of $\mathrm{Sp}(2g + 2, \mathbb{Z})$, so the image of ours does too. (One

can show that our extra generator doesn't enlarge the image of the representation.)

Since the image of H contains the level-2 congruence subgroup of $\mathrm{Sp}(2g+2, \mathbb{Z})$, all we have to show is that H surjects to $\mathrm{Sp}(2g+2, \mathbb{Z}/2)$. We did this in lemma 6. $\qquad\square$

Lemma 9. *If $g \geq 0$ and $3 \nmid N$, then the projection of K to any factor of $P_N = \prod_\Omega \mathrm{Sp}(2g + 2, \mathbb{Z}/N)$ is surjective.*

Proof. Follow the proof of lemma 6; the only modification needed is that depending on one's definition of a transvection, β_{b-1}^3 might not be one (e.g. if 3 is not a square in $\mathbb{Z}/N$). But regardless of this choice of definition, if $m \geq 1$ satisfies $3m \equiv 1 \pmod{N}$, then the cyclic group β_{b-1}^3 generates contains the transvection β_{b-1}^{3m}. $\qquad\square$

Proof of theorem 2: It suffices to prove that K surjects to P_N, and by the Chinese remainder theorem it suffices to treat the case where N is a prime power p^n. First we treat the case $N = p$. Under our hypothesis on g, $\mathrm{PSp}(2g + 2, \mathbb{Z}/p)$ is a nonabelian simple group. Then the argument for theorem 1 implies that K surjects to the central quotient $\prod_\Omega \mathrm{PSp}(2g+2, \mathbb{Z}/p)$ of P_p. Since $\mathrm{Sp}(2g+2, \mathbb{Z}/p)$ is a nonsplit extension of $\mathrm{PSp}(2g + 2, \mathbb{Z}/p)$, K surjects to P_p.

Now we suppose $N = p^n$ for $n > 1$. We write Γ for the level p^{n-1} congruence subgroup of $\mathrm{Sp}(2g+2, \mathbb{Z}/p^n)$ and assume inductively that K surjects to $P_{p^{n-1}}$. So we must show that $G_N \cap \prod_\Omega \Gamma$ is all of $\prod_\Omega \Gamma$. Now, Γ is an elementary abelian p-group, and the action of $\mathrm{Sp}(2g + 2, \mathbb{Z}/p^n)$ on it factors through $\mathrm{Sp}(2g + 2, \mathbb{Z}/p)$, whose action on Γ is equivalent to the adjoint action on $\mathfrak{sp}(2g + 2, \mathbb{Z}/p)$. First we suppose $p > 2$, so that this action is irreducible. Observe that the action of P_p on $\prod_\Omega \Gamma$ is by the direct sum of $|\Omega|$ many distinct irreducible representations of P_p. Since G_N surjects to P_p, $G_N \cap \prod_\Omega \Gamma$ is an invariant subspace, so it is the product of some of the factors of $\prod_\Omega \Gamma$. It also surjects to each factor, by lemma 9, so it must be the product of all of them. This finishes the proof for $p \neq 2$.

The same argument works for $p = 2$, even though Γ is no longer irreducible under $\mathrm{Sp}(2g + 2, \mathbb{Z}/2)$. The scalar matrix $1 + 2^{n-1}$ in $\Gamma \cong \mathfrak{sp}(2g + 2, \mathbb{Z}/2)$ is invariant, the quotient by the span of this vector is irreducible, and there is no invariant complement. This last property is key, because it implies that the only P_p-invariant subspace of $\prod_\Omega \Gamma$ that projects onto each factor is the whole product. So the argument still applies. $\qquad\square$

Now we explain the application of theorem 2 to Ellenberg's question. As in the previous section, we will indicate degrees of covers of $\mathbb{P}^1$ by

subscripts. Suppose $3 \nmid N$ and $C_{6N^2} \in \mathcal{E}_N$, i.e., C_{6N^2} is a Galois cover of $\mathbb{P}^1$ with Galois group $X_N = N^2{:}S_3$ and b branch points, and the small loops around them permute the sheets by involutions in X_N with nontrivial image in S_3. Then b must be even because the product of the b loops in $\mathbb{Z}/2 = S_3/3$ must be trivial. Analogously to section 1, we define C_6 as C_{6N^2}/N^2, C_2 as $C_6/3$ and the three C_3's as the quotients of C_6 by the 3 involutions in S_3. These are exactly the same covers we met in section 1 and they fit into a diagram similar to Figure 1.

We define the projectivization $\mathbb{P}V_N(C_6)$ as the set of direct summands $\mathbb{Z}/N$ of $V_N(C_6)$. The arguments of section 1, with $\mathbb{Z}/N$ in place of $\mathbb{Z}/2$, imply that C_{6N^2} corresponds to an element of $\mathbb{P}V_N(C_6)$, and that the fiber of $\mathcal{E}_N \to \mathcal{P}_b$ over $p \in \mathcal{P}_b$ is in bijection with the set of pairs (C_6, C_{6N^2}), where C_6 corresponds to an element of $\Omega = \mathbb{P}H^1(C_2; \mathbb{Z}/3) \cong \mathbb{P}^{b-3}(\mathbb{Z}/3)$ and C_{6N^2} to an element of $\mathbb{P}V_N(C_6)$. That is, the fiber is $\coprod_\Omega \mathbb{P}^{b-5}(\mathbb{Z}/N)$. It is clear that the action of G on this set is determined by its action on $\oplus_\Omega V(C_6)$, which is exactly the fiber of $\mathcal{V}_N$. Indeed, the action is given by projectivizing the action on each summand, so the monodromy group $\bar{G}_N$ is got from (2) by replacing Sp by PSp.

Proof of theorem 3: We have already explained why $\bar{G}_N$ is the quotient of G_N by the center of $K_N = \prod_\Omega \mathrm{Sp}(b-4, \mathbb{Z}/N)$, so all we have to do is show that the sequence splits. By the Chinese remainder theorem, it suffices to treat the case with N a prime power p^n. We appeal to a theorem of Gross and Kovács [13, Cor. 4.4] which describes the structure of extensions like (3) in terms of the stabilizer of one factor of the product. We fix $\omega \in \Omega$ and let $\bar{H}_N \subseteq \bar{G}_N$ be its stabilizer. Their result asserts that (3) splits if and only if

$$1 \to \prod_{\omega' \in \Omega} S \Big/ \prod_{\omega' \neq \omega} S \to \bar{H}_N \Big/ \prod_{\omega' \neq \omega} S \to \bar{H}_N \Big/ \prod_{\omega' \in \Omega} S \to 1,$$

does, where $S = \mathrm{PSp}(b-4, \mathbb{Z}/p^n)$. This sequence has the form

$$1 \to \mathrm{PSp}(b-4, \mathbb{Z}/p^n) \to ? \to 3{\cdot}3^{b-4}{:}\mathrm{Sp}(b-4, \mathbb{Z}/3) \to 1,$$

the right term being a maximal parabolic subgroup of $\mathrm{PSp}(b-2, \mathbb{Z}/3)$. Since the left term is centerless, the structure of the extension is given by the natural homomorphism from the right term to $\mathrm{Out}(S)$, which is solvable. Since the right term is perfect, this map is trivial, so the sequence splits, so (3) does too.

($\mathrm{Out}(S)$ is known exactly, cf. [16] for the case $b \geq 10$ and [1] for the case $b \geq 6$ with N odd. But it is much easier to see solvability than to work the group out exactly.) $\qquad\square$

Remark. Since we know $\bar{G}_N$, we recover the result of Biggers and Fried [5] that G is transitive on the fiber of $\mathcal{E}_N \to \mathcal{P}_b$, which is the same as the irreducibility of $\mathcal{E}_N$. On the other hand, when $N \neq 0$ one could use their result to prove an analogue of lemma 8 without relying on A'Campo's theorem. Namely, H acts on $\mathbb{P}V$ as $\mathrm{PSp}(V_N)$; one mimics the proof of lemma 5, using their transitivity result in place of Clebsch's. One can then use this to prove lemma 8 itself (for $N \neq 0$).

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